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TOTALLY REAL SUBMANIFOLDS OF A KAEHLERIAN MANIFOLD

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To Katsumi Nomizu on his fiftieth birthday

0. Introduction

The purpose of the present paper is first to establish, for a totally real submanifold of a Kaehlerian manifold, the equations of Gauss and Ricci which contain the Bochner curvature tensor of the ambient Kaehlerian manifold, the Weyl conformal curvature tensor and the second fundamental tensors of the submanifold, and then to prove the following three theorems.

Theorem 1. *Let M^n, $n \geq 4$, be a totally umbilical, totally real submanifold of a Kaehlerian manifold M^{2m} with vanishing Bochner curvature tensor. Then M^n is conformally flat.*

Theorem 2. *Let M^3 be a totally geodesic, totally real submanifold of a Kaehlerian manifold with vanishing Bochner curvature tensor. Then M^3 is conformally flat.*

Theorem 3. *Let M^n, $n \geq 4$, be a totally real submanifold of a Kaehlerian manifold M^{2n} with vanishing Bochner curvature tensor. If the second fundamental tensors of M^n commute, then M^n is conformally flat.*

Theorem 1 generalizes a theorem of Blair [1]:

Theorem A. *Let M^{2n}, $n \geq 4$, be a Kaehler manifold with vanishing Bochner curvature tensor, and let M^n be a totally geodesic, totally real submanifold of M^{2n}. Then M^n is conformally flat.*

Theorem 3 generalizes also the theorem of Blair in a different direction.

In § 1 we state some known results on Weyl and Bochner curvature tensors, which we need in the sequel. In § 2 we establish the equations of Gauss for a totally real submanifold of a Kaehlerian manifold and prove Theorem 1. In § 3 we prove Theorem 2. In § 4 we establish the equations of Ricci for a totally real submanifold M^n of a Kaehlerian manifold M^{2n} and prove Theorem 3.

1. Preliminaries

Let M^n, $n \geq 3$, be an n-dimensional Riemannian manifold of class C^∞ covered by a system of coordinate neighborhoods $\{U; x^h\}$, where and in the

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sequel the indices $h, i, j, k, \cdots$ run over the range $\{1', 2', \cdots, n'\}$, and let g_{ji}, ∇_j, $K_{kji}{}^h$, K_{ji} and K be the positive definite metric tensor, the operator of covariant differentiation with respect to the Christoffel symbols $\{{}_j{}^h{}_i\}$ formed with g_{ji}, the curvature tensor, the Ricci tensor and the scalar curvature of M^n respectively.

A change of metric $g_{ji} \to \rho^2 g_{ji}$, where ρ is a scalar function such that $\rho^2 > 0$, is called a conformal change of metric. The Weyl conformal curvature tensor defined by

$$C_{kji}{}^h = K_{kji}{}^h + \delta_k^h C_{ji} - \delta_j^h C_{ki} + C_k{}^h g_{ji} - C_j{}^h g_{ki} \tag{1.1}$$

is invariant under a conformal change of metric, where

$$C_{ji} = -\frac{1}{n-2} K_{ji} + \frac{1}{2(n-1)(n-2)} K g_{ji}\,, \qquad C_k{}^h = C_{kt} g^{th}\,, \tag{1.2}$$

g^{th} being the contravariant components of the metric tensor. For $n = 3$, $C_{kji}{}^h$ vanishes identically and the curvature tensor defined by

$$C_{kji} = \nabla_k C_{ji} - \nabla_j C_{ki} \tag{1.3}$$

is invariant under a conformal change of metric.

If the metric of M^n is conformal to that of a locally Euclidean space, then M^n is said to be conformally flat. It is well known that a conformally flat Riemannian manifold M^n is characterized, for $n > 3$, by the vanishing of the Weyl conformal curvature tensor $C_{kji}{}^h$ and, for $n = 3$, by the vanishing of the curvature tensor C_{kji}.

Bochner [2] (see also Yano and Bochner [12]) proved

Theorem B. *If a compact orientable conformally flat Riemannian manifold M^n has positive definite Ricci curvature, then we have $b_p = 0$ $(0 < p < n)$, where b_p denotes the p-th Betti number of the manifold.*

To obtain a theorem corresponding to Theorem B in a Kaehlerian manifold, Bochner [3] (see also Yano and Bochner [12]) introduced a Kaehler analogue of the Weyl conformal curvature tensor.

Let M^{2m}, $m \geq 2$, be a real $2m$-dimensional Kaehlerian manifold covered by a system of coordinate neighborhoods $\{V; y^\kappa\}$, where and in the sequel the indices $\kappa, \lambda, \mu, \nu, \cdots$ run over the range $\{1, 2, \cdots, 2m\}$, and let $g_{\mu\lambda}$, $F_\lambda{}^\kappa$, ∇_λ, $K_{\nu\mu\lambda}{}^\kappa$, $K_{\mu\lambda}$ and k be the positive definite Hermitian metric tensor, the complex structure tensor, the operator of covariant differentiation with respect to the Christoffel symbols $\{{}_\mu{}^\kappa{}_\lambda\}$ formed with $g_{\mu\lambda}$, the curvature tensor, the Ricci tensor and the scalar curvature of M^{2m} respectively. Then the Bochner curvature tensor is defined by

$$
\begin{aligned}
B_{\nu\mu\lambda}{}^{\kappa} = K_{\nu\mu\lambda}{}^{\kappa} &+ \delta_\nu^\kappa L_{\mu\lambda} - \delta_\mu^\kappa L_{\nu\lambda} + L_\nu{}^\kappa g_{\mu\lambda} - L_\mu{}^\kappa g_{\nu\lambda} + F_\nu{}^\kappa L'_{\mu\lambda} \\
&- F_\mu{}^\kappa L'_{\nu\lambda} + L'^{\kappa}_{\nu} F_{\mu\lambda} - L'^{\kappa}_{\mu} F_{\nu\lambda} - 2(F_{\nu\mu} L'^{\kappa}_{\lambda} + L'_{\nu\mu} F_\lambda{}^\kappa) ,
\end{aligned} \tag{1.4}
$$

where

$$
\begin{aligned}
L_{\mu\lambda} &= -\frac{1}{2(m+2)} K_{\mu\lambda} + \frac{1}{8(m+1)(m+2)} k g_{\mu\lambda} , \qquad L_\nu{}^\kappa = L_{\nu\alpha} g^{\alpha\kappa} , \\
L'_{\mu\lambda} &= -L_{\mu\alpha} F_\lambda{}^\alpha , \qquad L'_\nu{}^\kappa = L'_{\nu\alpha} g^{\alpha\kappa} ,
\end{aligned} \tag{1.5}
$$

and $F_{\mu\lambda} = F_\mu{}^\alpha g_{\alpha\lambda}$, $g^{\alpha\kappa}$ being the contravariant components of the metric tensor.

Bochner introduced this curvature tensor using a complex coordinate system. The tensor expression (1.4) of the Bochner curvature tensor in a real coordinate system has been given by Tachibana [7].

In the sequel, we need the following identity satisfied by the Bochner curvature tensor:

$$
\begin{aligned}
\nabla_\kappa B_{\nu\mu\lambda}{}^{\kappa} = -2m\Big[&\nabla_\nu L_{\mu\lambda} - \nabla_\mu L_{\nu\lambda} \\
&+ \frac{1}{8(m+1)(m+2)} (F_\nu{}^\kappa F_{\mu\lambda} - F_\mu{}^\kappa F_{\nu\lambda} - 2F_{\nu\mu} F_\lambda{}^\kappa) \nabla_\kappa k \Big] ,
\end{aligned} \tag{1.6}
$$

which was obtained by Tachibana [7].

Introducing this curvature tensor, Bochner [3] (see also Yano and Bochner [12]) proved

Theorem C. *If a compact Kaehlerian manifold M^{2m} with vanishing Bochner curvature tensor has positive definite Ricci tensor, then we have $b_{2p-1} = 0$, $b_{2p} = 1$ $(0 < p < m)$, where b_i denotes the i-th Betti number of the manifold.*

Concerning the analogy between the Weyl conformal curvature tensor and the Bochner curvature tensor, see Chen and Yano [5], Takagi and Watanabe [8], Yano [11] and Yano and Ishihara [13].

2. Equations of Gauss and the proof of Theorem 1

We assume that an n-dimensional Riemannian manifold M^n is isometrically immersed in a real $2m$-dimensional Kaehlerian manifold M^{2m}, and represent the immersion by $y^\kappa = y^\kappa(x^h)$. We then put $B_i{}^\kappa = \partial y^\kappa / \partial x^i$, and denote by $C_y{}^\kappa$ $2m - n$ mutually orthogonal unit normals to M^n, where and in the sequel the indices x, y, z run over the range $\{(n+1)', (n+2)', \cdots, (2m)'\}$. Thus we have $g_{ji} = g_{\mu\lambda} B_{ji}^{\mu\lambda}$, where $B_{ji}^{\mu\lambda} = B_j{}^\mu B_i{}^\lambda$, and the metric tensor in the normal bundle is given by $g_{zy} = g_{\mu\lambda} C_{zy}^{\mu\lambda}$, where $C_{zy}^{\mu\lambda} = C_z{}^\mu C_y{}^\lambda$.

Now, if the transform of the tangent space at each point of M^n by the complex structure tensor F is orthogonal to the tangent space, then the submanifold is said to be totally real (Chen and Ogiue [4], Ludden, Okumura and

Yano [6]). For a totally real submanifold M^n we have equations of the form

$$F_\lambda{}^\kappa B_i{}^\lambda = -f_i{}^x C_x{}^\kappa , \tag{2.1}$$

$$F_\lambda{}^\kappa C_y{}^\lambda = f_y{}^h B_h{}^\kappa + f_y{}^x C_x{}^\kappa, \tag{2.2}$$

from which follows

$$F_\lambda{}^\kappa B_i{}^\lambda B^h{}_\kappa = 0 \quad \text{or} \quad F_{\mu\lambda} B_{ji}^{\mu\lambda} = 0 , \tag{2.3}$$

where $B^h{}_\kappa = B_i{}^\lambda g^{ih} g_{\lambda\kappa}$.

Equations of Gauss and Weingarten for M^n of M^{2m} are respectively

$$\nabla_j B_i^\kappa = H_{ji}{}^x C_x{}^\kappa , \tag{2.4}$$

$$\nabla_j C_y{}^\kappa = -H_j{}^i{}_y B_i{}^\kappa, \tag{2.5}$$

where $H_{ji}{}^x$ are the second fundamental tensors of M^n with respect to the normals $C_x{}^\kappa$, $H_j{}^i{}_y = H_{jty} g^{ti}$ and $H_{jty} = H_{jt}{}^z g_{zy}$.

From (2.4) and (2.5) we can deduce equations of Gauss for a submanifold M^n of M^{2m}:

$$K_{kji}{}^h = K_{\nu\mu\lambda}{}^\kappa B_{kji\kappa}^{\nu\mu\lambda h} + H_k{}^h{}_x H_{ji}{}^x - H_j{}^h{}_x H_{ki}{}^x , \tag{2.6}$$

where $B_{kji\kappa}^{\nu\mu\lambda h} = B_k{}^\nu B_j{}^\mu B_i{}^\lambda B^h{}_\kappa$. Introduce the notation

$$M_{ji}{}^x = H_{ji}{}^x - H^x g_{ji} , \tag{2.7}$$

where $H^x = \dfrac{1}{n} g^{ts} H_{ts}{}^x$. The $M_{ji}{}^x$ are called conformal second fundamental tensors of M^n with respect to the normals $C_x{}^\kappa$. Indeed $M_{ji}{}^x C_x{}^\kappa$ is invariant under a conformal change of metric of the ambient manifold (cf. Yano [9]). We notice that $M_{ji}{}^x$ thus defined satisfies $g^{ji} M_{ji}{}^x = 0$. A submanifold M^n is totally umbilical if and only if $M_{ji}{}^x = 0$. Using $M_{ji}{}^x$ we rewrite (2.6) in the form

$$\begin{aligned} K_{kji}{}^h = {} & K_{\nu\mu\lambda}{}^\kappa B_{kji\kappa}^{\nu\mu\lambda h} + M_k{}^h{}_x M_{ji}{}^x - M_j{}^h{}_x M_{ki}{}^x + \delta_k^h M_{ji}{}^x H_x \\ & - \delta_j^h M_{ki}{}^x H_x + M_k{}^h{}_x H^x g_{ji} - M_j{}^h{}_x H^x g_{ki} \\ & + H_x H^x (\delta_k^h g_{ji} - \delta_j^h g_{ki}) , \end{aligned} \tag{2.8}$$

where $M_k{}^h{}_x = M_{ktx} g^{th}$, $M_{ktx} = M_{kt}{}^y g_{yx}$ and $H_x = H^y g_{yx}$.

Now transvecting (1.4) with $B_{kji\kappa}^{\nu\mu\lambda h}$ and using (2.3) for a totally real submanifold, we obtain

$$B_{\nu\mu\lambda}{}^\kappa B_{kji\kappa}^{\nu\mu\lambda h} = K_{\nu\mu\lambda}{}^\kappa B_{kji\kappa}^{\nu\mu\lambda h} + \delta_k^h L_{\mu\lambda} B_{ji}^{\mu\lambda} - \delta_j^h L_{\mu\lambda} B_{ki}^{\mu\lambda} + L_{\mu\lambda} B_{kt}^{\mu\lambda} g^{th} g_{ji} - L_{\mu\lambda} B_{jt}^{\mu\lambda} g^{th} g_{ki} ,$$

and consequently we can write (2.8) in the form:

$$
\begin{aligned}
K_{kji}{}^{h} = {} & B_{\nu\mu\lambda}{}^{\kappa} B_{kji\kappa}^{\nu\mu\lambda h} - \delta_k^h L_{\mu\lambda} B_{ji}^{\mu\lambda} + \delta_j^h L_{\mu\lambda} B_{ki}^{\mu\lambda} - L_{\mu\lambda} B_{kt}^{\mu\lambda} g^{th} g_{ji} \\
& + L_{\mu\lambda} B_{jt}^{\mu\lambda} g^{th} g_{ki} + M_k{}^h{}_x M_{ji}{}^x - M_j{}^h{}_x M_{ki}{}^x + \delta_k^h M_{ji}{}^x H_x \\
& - \delta_j^h M_{ki}{}^x H_x + M_k{}^h{}_x H^x g_{ji} - M_j{}^h{}_x H^x g_{ki} + H_x H^x (\delta_k^h g_{ji} - \delta_j^h g_{ki}) \ .
\end{aligned} \tag{2.9}
$$

From (2.9), contracting with respect to h and k and remembering $M_t{}^t{}_x = 0$, we find

$$
\begin{aligned}
K_{ji} = {} & B_{\nu\mu\lambda}{}^{\kappa} B_{\kappa}^{\nu} B_{ji}^{\mu\lambda} - (n-2) L_{\mu\lambda} B_{ji}^{\mu\lambda} - L_{\mu\lambda} B^{\mu\lambda} g_{ji} \\
& - M_j{}^t{}_x M_{ti}{}^x + (n-2) M_{ji}{}^x H_x + (n-1) H_x H^x g_{ji} \ ,
\end{aligned} \tag{2.10}
$$

where $B_{\kappa}^{\nu} = B_t{}^{\nu} B^t{}_{\kappa}$ and $B^{\mu\lambda} = B_{ji}^{\mu\lambda} g^{ji}$. Transvecting (2.10) with g^{ji} gives

$$
\begin{aligned}
L_{\mu\lambda} B^{\mu\lambda} = {} & -\frac{1}{2(n-1)} K + \frac{1}{2(n-1)} B_{\nu\mu\lambda}{}^{\kappa} B_{\kappa}^{\nu} B^{\mu\lambda} \\
& - \frac{1}{2(n-1)} M_s{}^t{}_x M_t{}^{sx} + \frac{n}{2} H_x H^x \ .
\end{aligned} \tag{2.11}
$$

Substituting (2.11) in (2.10), we find

$$
\begin{aligned}
L_{\mu\lambda} B_{ji}^{\mu\lambda} = {} & C_{ji} + \frac{1}{n-2} B_{\nu\mu\lambda}{}^{\kappa} B_{\kappa}^{\nu} B_{ji}^{\mu\lambda} - \frac{1}{2(n-1)(n-2)} B_{\nu\mu\lambda}{}^{\kappa} B_{\kappa}^{\nu} B^{\mu\lambda} g_{ji} \\
& - \frac{1}{n-2} M_j{}^t{}_x M_{ti}{}^x + \frac{1}{2(n-1)(n-2)} M_s{}^t{}_x M_t{}^{sx} g_{ji} \\
& + M_{ji}{}^x H_x + \frac{1}{2} H_x H^x g_{ji} \ .
\end{aligned} \tag{2.12}
$$

Substituting (2.12) in (2.9) yields

$$
\begin{aligned}
C_{kji}{}^{h} = {} & B_{\nu\mu\lambda}{}^{\kappa} B_{kji\kappa}^{\nu\mu\lambda h} - \frac{1}{n-2} \Big[\delta_k^h B_{\nu\mu\lambda}{}^{\kappa} B_{\kappa}^{\nu} B_{ji}^{\mu\lambda} - \delta_j^h B_{\nu\mu\lambda}{}^{\kappa} B_{\kappa}^{\lambda} B_{ki}^{\mu\lambda} \\
& \qquad + B_{\nu\mu\lambda}{}^{\kappa} B_{\kappa}^{\nu} B_{kt}^{\mu\lambda} g^{th} g_{ji} - B_{\nu\mu\lambda}{}^{\kappa} B_{\kappa}^{\nu} B_{jt}^{\mu\lambda} g^{th} g_{ki} \Big] \\
& + \frac{1}{(n-1)(n-2)} B_{\nu\mu\lambda}{}^{\kappa} B_{\kappa}^{\nu} B^{\mu\lambda} (\delta_k^h g_{ji} - \delta_j^h g_{ki}) \\
& + M_k{}^h{}_x M_{ji}{}^x - M_j{}^h{}_x M_{ki}{}^x \\
& + \frac{1}{n-2} [\delta_k^h M_j{}^t{}_x M_{ti}{}^x - \delta_j^h M_k{}^t{}_x M_{ti}{}^x \\
& \qquad + M_k{}^t{}_x M_t{}^{hx} g_{ji} - M_j{}^t{}_x M_t{}^{hx} g_{ki}] \\
& - \frac{1}{(n-1)(n-2)} M_s{}^t{}_x M_t{}^{sx} (\delta_k^h g_{ji} - \delta_j^h g_{ki}) \ .
\end{aligned} \tag{2.13}
$$

These are equations of Gauss for a totally real submanifold M^n of a Kaehlerian manifold M^{2m} (cf. Yano [9]).

From (2.13) we see that if the Bochner curvature tensor of the ambient Kaehlerian manifold vanishes, and the totally real submanifold M^n is totally umbilical, that is, $M_{ji}{}^x = 0$, then we have $C_{kji}{}^h = 0$, which gives the proof of Theorem 1.

3. Totally geodesic, totally real submanifold and the proof of Theorem 2

For a totally geodesic, totally real submanifold, from (2.12) we have

$$L_{\mu\lambda}B_{ji}^{\mu\lambda} = C_{ji} + \frac{1}{n-2}\left[B_{\nu\mu\lambda}{}^{\epsilon}B_{\epsilon}^{\nu}B_{ji}^{\mu\lambda} - \frac{1}{2(n-1)}B_{\nu\mu\lambda}{}^{\epsilon}B_{\epsilon}^{\nu}B^{\mu\lambda}g_{ji}\right]. \tag{3.1}$$

On the other hand, transvecting (1.6) with $B_{kji}^{\nu\mu\lambda} = B_k{}^\nu B_j{}^\mu B_i{}^\lambda$ and using the fact that M^n is totally geodesic and totally real, we find

$$(\nabla_\epsilon B_{\nu\mu\lambda}{}^{\epsilon})B_{kji}^{\nu\mu\lambda} = -2m[\nabla_k(L_{\mu\lambda}B_{ji}^{\mu\lambda}) - \nabla_j(L_{\mu\lambda}B_{ki}^{\mu\lambda})], \tag{3.2}$$

which, together with (3.1), implies

$$\begin{aligned}(\nabla_\epsilon B_{\nu\mu\lambda}{}^{\epsilon})B_{kji}^{\nu\mu\lambda} = -2m\Big[&\nabla_k C_{ji} - \nabla_j C_{ki}\\ &+ \frac{1}{n-2}\nabla_k\left(B_{\nu\mu\lambda}{}^{\epsilon}B_{\epsilon}^{\nu}B_{ji}^{\mu\lambda} - \frac{1}{2(n-1)}B_{\nu\mu\lambda}{}^{\epsilon}B_{\epsilon}^{\nu}B^{\mu\lambda}g_{ji}\right)\\ &- \frac{1}{n-2}\nabla_j\left(B_{\nu\mu\lambda}{}^{\epsilon}B_{\epsilon}^{\nu}B_{ki}^{\mu\lambda} - \frac{1}{2(n-1)}B_{\nu\mu\lambda}{}^{\epsilon}B_{\epsilon}^{\nu}B^{\mu\lambda}g_{ki}\right)\Big].\end{aligned} \tag{3.3}$$

Thus for a totally geodesic, totally real submanifold of a Kaehlerian manifold with vanishing Bochner curvature tensor, we have

$$\nabla_k C_{ji} - \nabla_j C_{ki} = 0,$$

which proves Theorem 2.

4. Equations of Ricci and the proof of Theorem 3

Let M^n, $n \geq 4$, be a totally real submanifold of a Kaehlerian manifold M^{2m}. Then we have (2.1) and (2.2). Since $F_{\mu\lambda} = -F_{\lambda\mu}$ and consequently $F_{\mu\lambda}B_i{}^\mu C_y{}^\lambda = -F_{\mu\lambda}C_y{}^\mu B_i{}^\lambda$, from (2.1) and (2.2) follows

$$f_{iy} = f_{yi}, \tag{4.1}$$

where $f_{iy} = f_i{}^z g_{zy}$ and $f_{yi} = f_y{}^j g_{ji}$.

Applying the complex structure tensor F to (2.1) and (2.2) and using these equations, we find

$$f_i{}^y f_y{}^h = \delta_i^h , \tag{4.2}$$

$$f_i{}^y f_y{}^x = 0 , \tag{4.3}$$

$$f_y{}^z f_z{}^h = 0 , \tag{4.4}$$

$$f_y{}^z f_z{}^x = -\delta_y^x + f_y{}^i f_i{}^x . \tag{4.5}$$

(4.4) and (4.5) show that if $f_y{}^x$ does not vanish, it defines an f-structure in the normal bundle (see Yano [10]).

Differentiating (2.1) and (2.2) covariantly over M^n, and using $\nabla_j F_\lambda{}^\kappa = B_j{}^\mu \nabla_\mu F_\lambda{}^\kappa = 0$ and the equations of Gauss and Weingarten, we find

$$H_{ji}{}^x f_x{}^h - H_j{}^h{}_x f_i{}^x = 0 , \tag{4.6}$$

$$\nabla_j f_i{}^x = -H_{ji}{}^y f_y{}^x , \tag{4.7}$$

$$\nabla_j f_y{}^h = H_j{}^h{}_x f_y{}^x , \tag{4.8}$$

$$\nabla_j f_y{}^x = H_j{}^i{}_y f_i{}^x - H_{ji}{}^x f_y{}^i . \tag{4.9}$$

When $n = m$, which will be assumed in the sequel, we obtain $f_i{}^x f_y{}^i = \delta_y^x$ from (4.2) and consequently $f_y{}^z f_z{}^x = 0$ from (4.5). Since $f_{yx} = f_y{}^z g_{zx} = F_{\mu\lambda} C_y{}^\mu C_x{}^\lambda$ is skew-symmetric, $f_y{}^x = 0$ and therefore (4.2), $\cdots$, (4.5) are reduced to

$$f_i{}^y f_y{}^h = \delta_i^h , \qquad f_y{}^i f_i{}^x = \delta_y^x . \tag{4.10}$$

$f_y{}^x = 0$ and (4.7) imply that $\nabla_j f_i{}^x = 0$ from which follows

$$K_{kjy}{}^x f_i{}^y = K_{kji}{}^h f_h{}^x , \tag{4.11}$$

where $K_{kjy}{}^x$ is the curvature tensor of the connection induced in the normal bundle. Thus

$$K_{kjyx} f_i{}^y f_h{}^x = K_{kjih} , \tag{4.12}$$

where $K_{kjyx} = K_{kjy}{}^z g_{zx}$ and $K_{kjih} = K_{kji}{}^t g_{th}$. (4.11) shows that $K_{kjy}{}^x = 0$ and $K_{kji}{}^h = 0$ are equivalent.

Now the equations of Ricci are

$$K_{kjyx} = K_{\nu\mu\lambda\kappa} B_{kj}^{\nu\mu} C_{yx}^{\lambda\kappa} - T_{kjyx} , \tag{4.13}$$

where $K_{\nu\mu\lambda\kappa} = K_{\nu\mu\lambda}{}^\alpha g_{\alpha\kappa}$ and

$$T_{kjyx} = H_k{}^t{}_y H_{jtx} - H_j{}^t{}_y H_{ktx} \, . \tag{4.14}$$

Write (1.4) in the form

$$\begin{aligned} B_{\nu\mu\lambda\kappa} = K_{\nu\mu\lambda\kappa} + g_{\nu\kappa}L_{\mu\lambda} - g_{\mu\kappa}L_{\nu\lambda} + L_{\nu\kappa}g_{\mu\lambda} - L_{\mu\kappa}g_{\nu\lambda} + F_{\nu\kappa}L'_{\mu\lambda} \\ - F_{\mu\kappa}L'_{\nu\lambda} + L'_{\nu\kappa}F_{\mu\lambda} - L'_{\mu\kappa}F_{\nu\lambda} - 2(F_{\nu\mu}L'_{\lambda\kappa} + L'_{\nu\mu}F_{\lambda\kappa}) \, , \end{aligned} \tag{4.15}$$

where $B_{\nu\mu\lambda\kappa} = B_{\nu\mu\lambda}{}^\alpha g_{\alpha\kappa}$, and transvect (4.15) with $B^{\nu\mu}_{kj}C^{\lambda\kappa}_{yx}$. Then using

$$F_{\mu\lambda}B^{\mu\lambda}_{ji} = 0 \, , \quad F_{\mu\lambda}B_j{}^\mu C_y{}^\lambda = -f_{jy} \, , \quad F_{\mu\lambda}C_y{}^\mu C_x{}^\lambda = 0 \tag{4.16}$$

and $L'_{\mu\lambda} = -L_{\mu\alpha}F_\lambda{}^\alpha$, we find

$$\begin{aligned} B_{\nu\mu\lambda\kappa}B^{\nu\mu}_{kj}C^{\lambda\kappa}_{yx} = K_{\nu\mu\lambda\kappa}B^{\nu\mu}_{kj}C^{\lambda\kappa}_{yx} + f_{kx}L_{\mu\lambda}B^{\mu\lambda}_{jt}f_y{}^t - f_{jx}L_{\mu\lambda}B^{\mu\lambda}_{kt}f_y{}^t \\ + L_{\mu\lambda}B^{\mu\lambda}_{kt}f_x{}^t f_{jy} - L_{\mu\lambda}B^{\mu\lambda}_{jt}f_x{}^t f_{ky} \, . \end{aligned} \tag{4.17}$$

Thus from (4.13) and (4.17) follows

$$\begin{aligned} K_{kjyx} = B_{\nu\mu\lambda\kappa}B^{\nu\mu}_{kj}C^{\lambda\kappa}_{yx} - f_{kx}L_{\mu\lambda}B^{\mu\lambda}_{jt}f_y{}^t + f_{jx}L_{\mu\lambda}B^{\mu\lambda}_{kt}f_y{}^t \\ - L_{\mu\lambda}B^{\mu\lambda}_{kt}f_x{}^t f_{jy} + L_{\mu\lambda}B^{\mu\lambda}_{jt}f_x{}^t f_{ky} - T_{kjyx} \, , \end{aligned} \tag{4.18}$$

which and (4.12) imply

$$\begin{aligned} K_{kjih} = B_{\nu\mu\lambda\kappa}B^{\nu\mu}_{kj}C^{\lambda\kappa}_{yx}f_i{}^y f_h{}^x - g_{kh}L_{\mu\lambda}B^{\mu\lambda}_{ji} + g_{jh}L_{\mu\lambda}B^{\mu\lambda}_{ki} \\ - L_{\mu\lambda}B^{\mu\lambda}_{kh}g_{ji} + L_{\mu\lambda}B^{\mu\lambda}_{jh}g_{ki} - T_{kjyx}f_i{}^y f_h{}^x \, . \end{aligned} \tag{4.19}$$

Transvecting (4.19) with g^{kh} gives

$$K_{ji} = -B_{\nu\mu\lambda\kappa}B^{\nu\mu}_{js}C^{\lambda\kappa}_{yx}f_i{}^y f^{sx} - (n-2)L_{\mu\lambda}B^{\mu\lambda}_{ji} - L_{\mu\lambda}B^{\mu\lambda}g_{ji} + T_{jsyx}f_i{}^y f^{sx} \, , \tag{4.20}$$

where $f^{sx} = f_t{}^x g^{ts}$, and transvecting (4.20) with g^{ji} gives

$$L_{\mu\lambda}B^{\mu\lambda} = -\frac{1}{2(n-1)}(K + B - T) \, , \tag{4.21}$$

where

$$B = B_{\nu\mu\lambda\kappa}B^{\nu\mu}_{ts}C^{\lambda\kappa}_{yx}f^{ty}f^{sx} \, , \qquad T = T_{tsyx}f^{ty}f^{sx} \, .$$

Substituting (4.21) in (4.20) we find

$$L_{\mu\lambda}B^{\mu\lambda}_{ji} = C_{ji} - B_{ji} - T_{ji} \, , \tag{4.22}$$

where

$$B_{ji} = -\frac{1}{n-2}B_{\nu\mu\lambda\kappa}B^{\nu\mu\lambda\kappa}_{jtyx}f_i{}^y f^{tx} + \frac{1}{2(n-1)(n-2)}Bg_{ji} ,$$

$$T_{ji} = +\frac{1}{n-2}T_{jsyx}f_i{}^y f^{sx} - \frac{1}{2(n-1)(n-2)}Tg_{ji} .$$

Substituting (4.22) in (4.19) we obtain

$$\begin{aligned}C_{kjih} = B_{\nu\mu\lambda\kappa}B^{\nu\mu}_{kj}C^{\lambda\kappa}_{yx}f_i{}^y f_h{}^x + g_{kh}B_{ji} - g_{jh}B_{ki} + B_{kh}g_{ji} - B_{jh}g_{ki} \\ + T_{kjyx}f_i{}^y f_h{}^x + g_{kh}T_{ji} - g_{jh}T_{ki} + T_{kh}g_{ji} - T_{jh}g_{ki} ,\end{aligned} \tag{4.23}$$

where $C_{kjih} = C_{kji}{}^t g_{th}$. These are the equations of Ricci for a totally real submanifold M^n of a Kaehlerian manifold M^{2n}.

From the equations of Ricci it follows that if $B_{\nu\mu\lambda\kappa} = 0$ and $T_{kjyx} = 0$, then $C_{kjih} = 0$, which proves Theorem 3.

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